123. THE POWER REQUIRED FOR THE THERMODYNAMIC HEATING OF BUILDINGS*.

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SOLUTION OF A PROBLEM.

(*a*) What horse-power would be required to supply a building with 1 lb. of air per second, heated mechanically from 50° to 80° Fahrenheit? Compare the fuel that an engine producing this effect as $\frac{1}{10}$ of the equivalent of the heat of combustion would consume, with that which would be required to heat directly the same quantity of air.

(*b*) Explain how this effect may be produced with perfect economy by operating on the air itself to change its temperature, and give dimensions &c. of an apparatus that may be convenient for the purpose.

(*c*) Show how the same apparatus may be adapted to give a supply of cooled air.

Ex. Let it be required to supply a building with 1 lb. of air per second, cooled from 80° to 50° Fahrenheit. Determine the horse-power wanted to work the apparatus in this case.

(*a*) Instead of heating the air directly, we can produce the required effect more economically by means of a perfect thermo-dynamic engine; and it is easy to show that this is the most economical way. We will consider the air heated pound by pound, and sent into the building at the end of the heating process. Generally, let T be the temperature of the unheated air, S the temperature to which we wish it heated. T, being the temperature of air, water, &c. external to the building, will be the temperature of our refrigerator; the pound of air to be heated will be our source (nominally), and by working the engine backwards instead of taking away, we will give heat to the source.

* [See "On the Economy of the Heating or Cooling of Buildings by means of Currents of Air," *Glasgow Phil. Soc. Proc.* Dec. 1852, reprinted, with an addition referring to Coleman's refrigerating process, in *Math. and Phys. Papers*, Vol. I. pp. 515—520.]

If a be the specific heat of air, adt units will be required to raise the temperature of the pound of air from t to $t+dt$, and the work which must be spent to supply this will be

$$Jadt\frac{t-T}{t+E^{-1}}*.$$

Let the whole work spent upon the pound of air be denoted by W; then we have

$$W=Ja\int_T^S\frac{t-T}{t+E^{-1}}\,dt;$$

whence

$$W=Ja\left\{(S-T)-\left(T+E^{-1}\right)\log\frac{S+E^{-1}}{T+E^{-1}}\right\}.$$

Ex. $S=80°$, $T=50°$, Fahrenheit.

As E is usually given with reference to units Centigrade, we prefer reducing to that scale.

$$S=26°\,66' \text{ and } T=10° \text{ Cent.}, E=\cdot 00366.$$

$$E^{-1}=273\cdot 2240437,\ a=\cdot 24,\ J=1390.$$

$$W=1390\times\cdot 24\left\{16\cdot 66'-283\cdot 22404\log\frac{299\cdot 89071}{283\cdot 22404}\right\}$$

$$=1390\times\cdot 24\,\{16\cdot 66'-16\cdot 194713\}$$

$$=157\cdot 4438.$$

As one pound of air is heated per second, the H.P. of the engine will be got by dividing this by 550, so that

H.P. of engine $=\cdot 28626$.

If an engine (probably a steam-engine) be employed to drive the heating machine, and economise only $\frac{1}{10}$th of the fuel, the fuel must have evolved $10\,W/J$ units. To heat the pound directly $a(S-T)$ units must be supplied, and $\dfrac{10\,W/J}{a(S-T)}\times 100$ gives the percentage. In the particular case we have been considering, we find that to heat the air by means of an engine economising $\frac{1}{10}$ would require $\frac{4\cdot 7195}{16\cdot 6'}\times 100$, or $28\cdot 317$ per cent. of the fuel required for direct heating.

* For the formulas regarding the duty of a perfect engine, and the mechanical value of each of its cycles of operations, constantly to be employed in these solutions, we refer to a paper by Professor W. Thomson in the *Philosophical Transactions*, and which appears in the present number of this *Journal*.

(*b*) Conceive two double-stroke cylinders connected by tubes and valves in some convenient way, with a reservoir between them. Conceive the one to be made of perfectly conducting matter, so that there shall be no difference in temperature between internal and external air (practically this may be approximated to by immersion in running water); the other cylinder must, on the contrary, be perfectly non-conducting. The pressure in the reservoir being kept at an amount depending on the required heating effect, air is admitted (doing work as it enters) by the former, which we shall call the ingress cylinder, and is not allowed to cool below atmospheric temperature. It is pumped out by the latter, called the egress cylinder, and so heated by compression to the required temperature.

After these very general explanations, we proceed to mention more particularly the *details* of this process.

Let p', t' be respectively the atmospheric pressure and temperature, v' the volume of one pound of air under pressure p' and at temperature t', t the temperature to which we wish the air to be raised, p the pressure such that, if air under it be compressed to pressure p', the temperature will rise from t' to t, v the volume of one pound of air under pressure p and at temperature t', v_1 the volume of air under pressure p' and at temperature t. Now, by Poisson's formula and by the gaseous laws we have

$$\text{(A)} \qquad \frac{E^{-1}+t'}{E^{-1}+t}=\left(\frac{p}{p'}\right)^{\frac{K-1}{K}}, \qquad p=p'\left\{\frac{E^{-1}+t'}{E^{-1}+t}\right\}^{\frac{K}{K-1}},$$

from which p may be determined. p must be, moreover, the pressure in the reservoir, as will afterwards appear.

Volume of Cylinder. As the apparatus has to supply one pound of air per second, it will be convenient to suppose the cylinders of such a size, as to contain one pound of air at pressure p and temperature t'.

Operations in Ingress Cylinder. Suppose the piston at the top of its stroke, and the lower part of the cylinder connected with the reservoir, and consequently filled with air at pressure p and temperature t'. Then external air admitted above the piston will push it down ($p' > p$). In this the first part of the stroke, admit so much air that, when secondly it is allowed to expand at constant temperature t', we will have reached the end of the stroke by the time that the pressure has fallen to p. The lower part of the cylinder having been connected with the reservoir, has given to the latter the pound of air it contained; and at the end of the down-stroke the upper part is filled with air ready to be sent in by the up-stroke.

In this operation it is plain that we obtain mechanical effect, and we will naturally spend it, in helping to pump the air out by the egress cylinder.

Operations in Egress Cylinder. Suppose, as before, the piston at the top of its stroke, and the cylinder filled with air at pressure p and temperature t'. During the whole stroke you allow air from the reservoir to enter above the piston. During the first part of the stroke you compress the air below the piston, until the pressure becomes p' and the temperature consequently t. Then expel this heated air into the building or whatever place you wish to heat.

Estimate of total work spent.

(1) In ingress cylinder:

mechanical effect obtained during the first part of the stroke

$$=(p'-p)\,v';$$

mechanical effect obtained during the second part

$$=p'v'\log\frac{v}{v'}-p\,(v-v').$$

(B) Whole gain in ingress cylinder

$$=p'v'\log\frac{v}{v'}.$$

(2) In egress cylinder:

mechanical effect spent during compression

$$=\frac{pv}{K-1}\left\{\left(\frac{v}{v_1}\right)^{K-1}-1\right\}-p\,(v-v_1)$$

work spent during expulsion

$$=p'v_1-pv_1:$$

but

$$\frac{p'v_1}{pv}=\frac{E^{-1}+t}{E^{-1}+t'}=\left(\frac{v}{v_1}\right)^{K-1}.$$

(C) Whence whole work spent in egress cylinder

$$=\frac{pv}{K-1}\left\{\frac{E^{-1}+t}{E^{-1}+t'}-1\right\}+pv\left\{\frac{E^{-1}+t}{E^{-1}+t'}\right\}-pv,$$

$$=p'v'\frac{K}{K-1}\frac{t-t'}{E^{-1}+t'},$$

since $pv=p'v'$.

(D) Amount of work spent in both

$$=p'v'\frac{K}{K-1}\frac{t-t'}{E^{-1}+t'}-p'v'\log\frac{v}{v'}.*$$

Ratios of expansion:

in the first cylinder,

$$\frac{v'}{v}=\frac{p}{p'}=\left\{\frac{E^{-1}+t'}{E^{-1}+t}\right\}^{\frac{K}{K-1}},$$

in the second cylinder,

$$\frac{v_1}{v}=\left\{\frac{E^{-1}+t'}{E^{-1}+t}\right\}^{\frac{1}{K-1}}.$$

A slight consideration will shew, that the rates of the cylinders must be the same if we consider them as of the same size, and as each contains one pound of air at pressure p and temperature t', the rate will evidently be 30 double strokes per minute.

Let h be the height of the cylinder, and r the radius of the base; then volume of cylinder $=v=\pi r^2 h$, and if V_0 be the volume of a pound of air under pressure p' and at 0° Cent.,

$$\text{(E)}\qquad v'=V_0(1+Et'),$$

$$v=\frac{p'}{p}V_0(1+Et').$$

* Modifying this by means of the formulas

$$\frac{v}{v'}=\left\{\frac{E^{-1}+t}{E^{-1}+t'}\right\}^{\frac{K}{K-1}}\text{ and (E)},$$

we find the following as equivalent:

$$\frac{K}{K-1}Ep'V_0\left\{(t-t')-(t'+E^{-1})\log\frac{t+E^{-1}}{t'+E^{-1}}\right\};$$

which becomes identical with the expression given in division (*a*) when we substitute for $\frac{K}{K-1}Ep'V_0$ the value Ja, which it must have in consequence of the relation between the specific heats of air and the mechanical equivalent of the thermal unit established in another paper in this number of the *Journal*.

The preceding formulas give us the means of calculating readily the most useful results.

We will take as an example, to supply a building with one pound of air heated mechanically from 50° to 80° Fahr. (solved before, in question (*a*)). We have then $t' = 10°$ C., $t = 26{\cdot}6'$ C., $E = {\cdot}00366$, $1/E = 273{\cdot}22404$, $p' = 2114$ pounds per square foot. $K = 1{\cdot}41$. Then, by (A),

$$p = 2114\left(\frac{283{\cdot}22404}{299{\cdot}89071}\right)^{\frac{141}{41}}$$

$$= 1736{\cdot}6189 = 2114 \times {\cdot}8214848$$

$$= 2114 \times \frac{1}{1{\cdot}217308}.$$

All these forms are useful.

Volume of cylinder.

$$v' = V_0(1 + 10E)$$

$$= 12{\cdot}383 \times 1{\cdot}0366 = 12{\cdot}836218.$$

$$pv = p'v'. \quad v = \frac{p'}{p}v' = 1{\cdot}217308 \times 12{\cdot}836218.$$

Volume of cylinder

$$= v = 15{\cdot}6256.$$

A practically useful height of cylinder might be 4 feet, the corresponding diameter to which is 2·2302 feet.

Again, we have, amount of work spent in heating in this way one pound of air (D)

$$= \frac{K}{K-1}p'v'\left\{\frac{E^{-1}+t}{E^{-1}+t'} - 1\right\} - p'v'\log\frac{v}{v'}$$

$$= \frac{141}{41} \times 2114 \times 12{\cdot}836\left\{\frac{299{\cdot}89071}{283{\cdot}22404} - 1\right\}$$

$$- 2114 \times 12{\cdot}836 \log 1{\cdot}217308$$

$$= 5491{\cdot}54 - 5336{\cdot}03$$

$$= 155{\cdot}51 \text{ estimated in foot pounds.}$$

As one pound must be supplied per second, H.P. of engine required to drive the apparatus $= {\cdot}2827$. This result ought, inasmuch as this apparatus possesses all the qualifications of a perfect engine, to be identical with the answer found in division (*a*) of this problem; we however find a difference of ·0034 between the two, due to the circumstance that the number ·24 which we employed as the value of the specific heat of air in the previous solution, also 1·41 for K in this, are only approximately true; but the true H.P. to two significant figures is ·28.

Ratios of Expansion. In first cylinder

$$v'/v = \cdot 8214848,$$

so that 3·2859 feet of the stroke passed while air was being admitted at pressure 2114, and ·71406 feet in allowing this to expand to pressure of receiver.

In second cylinder

$$v_1/v = \cdot 869825,$$

or ·5207 feet of the stroke was spent in compressing the air from pressure p to pressure p' or 2114, the remaining 3·4793 feet in expelling it.

(*c*) The first suggestion, we believe, of an apparatus for cooling buildings by compressing air, was to pump in air into a reservoir and allow it to cool to the temperature of the atmosphere, on the supposition that if then allowed to rush out by means of a stopcock, it would, in consequence of the expansion, fall in temperature. Unfortunately however for this scheme, it has been found that there is only an almost imperceptible depression of temperature (after motion ceases in the air) due to a want of perfect rigour in Mayer's hypothesis. The friction of the air in the orifice &c. almost entirely compensates for the cold of expansion.

The apparatus described in (*b*) can, however, be very simply applied.

Instead of allowing the air to rush out, and thus heat itself by friction, let it out slowly, and make it work a piston in a double-stroke cylinder, and we shall not only obtain the full benefit of the cold of expansion, but also gain so much work as to make the H.P. of the engine required to drive the apparatus a mere trifle.

The working of the apparatus, however, will not be so simple as in the last case, for as we are to use the same apparatus, we cannot make the cylinders hold one pound of air, and cannot even have the pistons moving at the same rate.

Let p' and t' be the atmospheric pressure and temperature respectively, v' the volume of either cylinder, t the temperature of the cooled air, p the pressure in the receiver, which will be such that if air at pressure p and temperature t' be allowed to expand to pressure p' the temperature will become t, v the volume under pressure p of a quantity of air, which under pressure p' would fill the cylinder, there being no change of temperature, v_1 volume under pressure p and temperature t', of a quantity of air which would fill the cylinder under pressure p' and at temperature t.

$$\text{(A}'\text{)} \qquad \frac{p}{p'} = \left\{\frac{E^{-1}+t'}{E^{-1}+t}\right\}^{\frac{K}{K-1}}.$$

Operations in Ingress Cylinder. Suppose the piston at the top of its stroke, the cylinder full of air at ordinary pressure. Admitting external air above the piston, push the piston down until the air below is compressed to pressure p, the temperature being kept constant; and then send this compressed air into the reservoir.

Operations in Egress Cylinder. In the first part of the stroke allow so much air to enter the cylinder from the reservoir, that if allowed to expand in the remaining part of the stroke, the pressure and temperature at the end will be p' and t respectively.

In Ingress Cylinder. Work spent in compressing air from p' to p (temperature constant)

$$= p'v' \log \frac{v'}{v} - p'(v'-v).$$

Work spent in sending the air into the receiver

$$= pv - p'v.$$

(B′) Total expenditure per stroke in first cylinder

$$= p'v' \log \frac{v'}{v}.$$

In Egress Cylinder. Mechanical effect gained in partially filling the cylinder from reservoir

$$= pv_1 - p'v_1.$$

Mechanical effect gained during the rest of the stroke

$$=\frac{p'v'}{K-1}\left\{\left(\frac{v'}{v_1}\right)^{K-1}-1\right\}-p'(v'-v_1).$$

(C′) Total gain per single stroke

$$=\frac{p'v'}{K-1}\left\{\left(\frac{v'}{v_1}\right)^{K-1}-1\right\}+pv_1-p'v'$$

$$=\frac{p'v'}{K-1}\left\{\frac{E^{-1}+t'}{E^{-1}+t}-1\right\}+p'v'\left\{\frac{E^{-1}+t'}{E^{-1}+t}-1\right\}$$

$$=p'v'\frac{K}{K-1}\frac{t'-t}{E^{-1}+t}.$$

Ratios of Expansion. In ingress cylinder,

$$\frac{v}{v'}=\frac{p'}{p}=\left\{\frac{E^{-1}+t}{E^{-1}+t'}\right\}^{\frac{K}{K-1}}.$$

In egress cylinder,

$$\frac{v_1}{v'}=\left\{\frac{E^{-1}+t}{E^{-1}+t'}\right\}^{\frac{1}{K-1}}.$$

It would be a matter of no difficulty to give generally the number of pounds each cylinder would contain, and also the rate of each piston, but it would only tend to confusion of symbols, as we should have to take in the p's, t's, and v's of (*b*), as well as those of this case; we will give these details in the example, which is

$t'=80°$ Fahrenheit $=26{\cdot}6'$ Cent., $t=50°$ Fahrenheit $=10$ Cent., p' of course 2114, height of cylinder, as before, 4 feet, and consequently, diameter 2·2302 feet.

From (A′) we have $p=2114\times 1{\cdot}217308$ pounds per square foot.

In (*b*) we found volume of cylinder $=15{\cdot}6256=v'$, according to our present notation.

From (B′) we have total expenditure per single stroke in ingress cylinder

$$=p'v'\log\frac{v'}{v}=p'v'\log\frac{p}{p'}$$

$=1{\cdot}217308\times$ {the gain in ingress cylinder in (*b*)}

$=6495{\cdot}59.$

From (C′) we have total gain per single stroke in egress cylinder

$$= 1{\cdot}217308 \times \{\text{loss in egress cylinder in } (b)\}$$
$$= 6684{\cdot}9.$$

Ratios of Expansion. In ingress cylinder

$$v/v' = {\cdot}8214848,$$

and in egress

$$v_1/v' = {\cdot}869825,$$

as in the example attached to (b).

Let χ be the number of pounds the first cylinder contains at pressure p' (2114), and at temperature t' (26·6′ Cent.).

Volume of χ pounds $= 12{\cdot}383\chi\,(1 + E \times 26{\cdot}6) =$ volume of cylinder $= 12{\cdot}383\,(1 + E \times 10) \times 1{\cdot}217308$, as we found in ($b$); hence

$$\chi = \frac{283{\cdot}22404}{299{\cdot}89071}\,1{\cdot}217308 = 1{\cdot}149655.$$

Number of single strokes per second

$$= \chi^{-1} = {\cdot}8698348.$$

Number of double strokes per minute

$$= 26{\cdot}095044.$$

In the second cylinder, let χ' be the number of pounds contained by it at p' (2114) and t (10^c Cent.), we easily obtain

$$\chi' = 1{\cdot}217308.$$

Number of strokes per second

$$= \chi'^{-1} = {\cdot}8214848.$$

Number of double strokes per minute

$$= 24{\cdot}644544.$$

Whole work spent per second in ingress cylinder

$$= 6495{\cdot}59/\chi = 5650{\cdot}036.$$

Whole mechanical effect gained per second in egress cylinder

$$= 6684{\cdot}9/\chi' = 5491{\cdot}54.$$

Total motive power required

$$= 5650{\cdot}036 - 5491{\cdot}54 = 158{\cdot}496 \text{ ft. pounds per second.}$$

Hence, H.P. of engine required to drive the apparatus $= {\cdot}288$.