A Simple Recursive Digital Filter

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To learn a subject, it helps to start with the simplest possible examples first.
Once you understand the simple, the complex becomes easy. A good example of
this is the case of the Hydrogen atom in physics. Without having such a simple
atom to study, much of the development of atomic physics would probably not
have been possible.

The goal here is to look at the equivalent of the Hydrogen atom for recursive
digital filters. We will look at the simplest possible recursive filter, a first order
filter with constant coefficients. Even though the filter is simple, it is quite
useful in practice, especially when two or more of them are combined in various
ways. We will show how two of these filters can be combined to produce a
second order bandpass filter.

In what follows it is assumed that you know what the frequency response
of a filter means and how an equation for the frequency response can be gotten
from the recursion equation for the filter. As a quick reminder, if the input to
a linear time invariant filter is

$$f[n] = e^{in\Omega} \tag{1}$$

then the output will be

$$y[n] = e^{in\Omega}H(\Omega) \tag{2}$$

where the function $H(\Omega)$ is the frequency response of the filter. Why do we care
about the response of a filter to this particular type of input? Well, any general
input, $f[n]$ (at least the ones normally involved in digital filtering applications)
can be expressed as a linear combination of these complex exponentials

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\Omega)e^{in\Omega}d\Omega \tag{3}$$

where $F(\Omega)$ is the Fourier transform of $f[n]$

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-in\Omega} \tag{4}$$

So the output is then simply

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega)F(\Omega)e^{in\Omega}d\Omega \tag{5}$$

For more background on this see **Fourier Transform of a Sampled Signal** (http://www.exstrom.com/journal/sigproc/ftsampled.pdf).

It should be noted however that this simple relationship where the input of eq. 1 produces the output of eq. 2 is only true if the input started at $n = -\infty$ and the filter has been on for all time in the past. In practice this is of course never true. The input and the filtering start at some time designated as $n = 0$. In this case the output is still given by eq. 2 but now the response function $H(\Omega)$ is a function of n also

$$y[n] = e^{in\Omega}H(n,\Omega) \tag{6}$$

This is the startup response of the filter wheras eq. 2 is the long term or steady state response. As n becomes large, $H(n,\Omega)$ becomes equal to $H(\Omega)$.

$$\lim_{n\to\infty} H(n,\Omega) = H(\Omega) \tag{7}$$

So eq. 2 is a good approximation to eq. 6 when n is large. Only the steady state frequency response will be considered in what follows. Now on to the filters.

The simplest possible recursive filter that you can imagine is the first order filter whose recursion equation is

$$y[n] - cy[n-1] = f[n] \tag{8}$$

This filter is the digital equivalent of a simple RC or RL analog filter. In the analog case the RC or RL components can be configured for either high or low pass operation. The same can be done with this digital filter by changing the sign of the coefficient c. For positive c we have a lowpass filter while for negative c we have a highpass filter. But we can go a step further and make c complex. Specifically we will define c as follows

$$c = re^{i\theta} \tag{9}$$

The frequency response of the filter is then given by

$$H(\Omega) = \frac{1}{1 - re^{i\theta}e^{-i\Omega}} \tag{10}$$

and the magnitude is

$$|H(\Omega)| = \frac{1}{\sqrt{1 - 2r\cos(\theta - \Omega) + r^2}} \tag{11}$$

which has a maximum at $\Omega = \theta$.
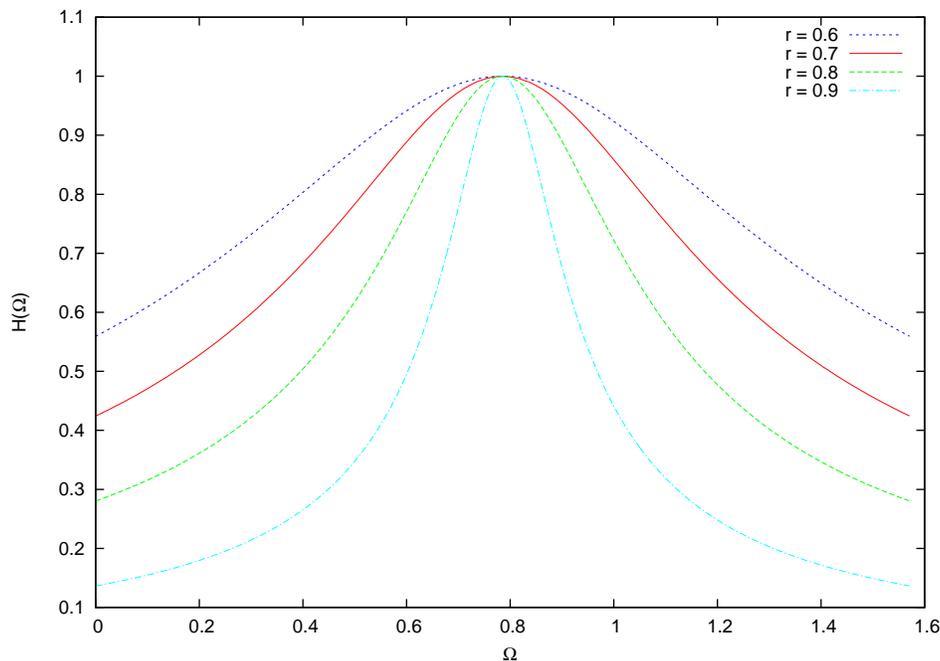
Figure 1: Eq. 10 normalized for $\theta = \pi/4$ and $r = [0.6, 0.7, 0.8, 0.9]$.

$$|H(\theta)| = \frac{1}{|1 - r|} \tag{12}$$

Therefore to normalize $H(\Omega)$ it should be multiplied by $|1 - r|$. Figure 1 shows a normalized plot of $H(\Omega)$ for $\theta = \pi/4$ and $r = [0.6, 0.7, 0.8, 0.9]$. Notice that the response indicates that this is a bandpass filter centered at $\Omega = \pi/4$ and that the r parameter determines the bandwidth.

This is not something you can do with a first order analog filter. You can not combine one resistor and one capacitor or one resistor and one inductor so as to make a bandpass filter. We managed to turn a first order digital filter into a bandpass filter by making the filter coefficient complex. It will become clear below however, that by making the coefficient complex we really have a second order filter in disguise.

By now you may be thinking: "If the filter coefficient is complex won't the output also be complex?". The answer is of course yes. If the input is real, which in most cases it will be, then the output will be complex if $0 < \theta < \pi$. So we need to turn this complex output into a real number. How do we do it and what does the result mean?

One possibility is to just use the real part of the output. Mathematically this means performing the following operation

$$\frac{y[n] + y^*[n]}{2} \tag{13}$$

But what we are doing here is really just equivalent to adding the output of two of our first order filters, one with $c = re^{i\theta}$, the other with $c = re^{-i\theta}$, and then multiplying the result by $1/2$. What does this mean in terms of frequency response? If you add the output of two filters, the response of the result is equal to the sum of the responses of the two filters. For this case then, the equivalent frequency response is given by:

$$
\begin{aligned}
H(\Omega) &= \frac{1}{2}\left(\frac{1}{1 - re^{i\theta}e^{-i\Omega}} + \frac{1}{1 - re^{-i\theta}e^{-i\Omega}}\right) \\
&= \frac{1 - r\cos(\theta)e^{-i\Omega}}{1 - 2r\cos(\theta)e^{-i\Omega} + r^2 e^{-i2\Omega}}
\end{aligned} \tag{14}
$$

which can recognized as the frequency response of a second order filter with the following recursion equation:

$$y[n] - 2r\cos(\theta)y[n-1] + r^2 y[n-2] = f[n] - r\cos(\theta)f[n-1] \tag{15}$$

This means that by taking just the real part of the output of our first order filter we really get the frequency response of a second order filter. The normalized response is shown in figure 2 for $\theta = \pi/4$ and $r = [0.6, 0.7, 0.8, 0.9]$.
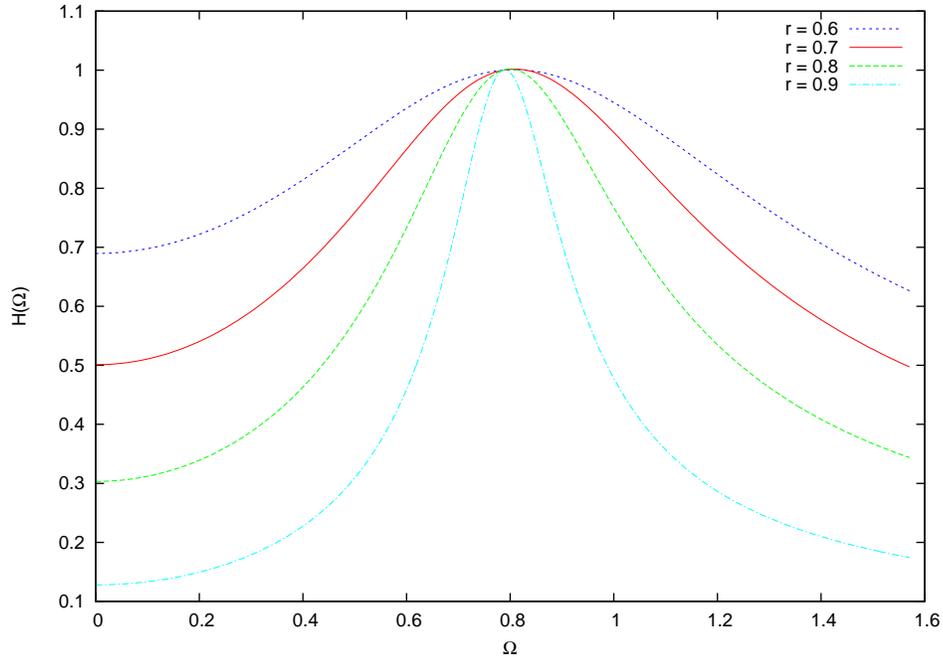
Figure 2: Eq. 14 normalized for $\theta = \pi/4$ and $r = [0.6, 0.7, 0.8, 0.9]$.

Notice that the response is still that of a bandpass filter with a maximum at $\Omega = \theta$ and a bandwidth controlled by the value of r. The peak magnitude that is used to normalize the response is:

$$|H(\theta)| = \frac{1}{|1-r|}\sqrt{\frac{1 - 2r\cos^2(\theta) + r^2\cos^2(\theta)}{1 - 2r\cos(2\theta) + r^2}} \qquad (16)$$

Now another way to get a real number from our filter is to just use the imaginary part of the output. Mathematically this means performing the following operation

$$\frac{y[n] - y^*[n]}{2i} \qquad (17)$$

This is similar to the previous case but here we are subtracting the output of two filters and multiplying the result by $1/2i$. To find out what this does to the overall frequency response we perform the same operation on the responses of the two filters. The equivalent response is then

$$\begin{aligned} H(\Omega) &= \frac{1}{2i}\left(\frac{1}{1 - re^{i\theta}e^{-i\Omega}} - \frac{1}{1 - re^{-i\theta}e^{-i\Omega}}\right) \qquad (18) \\ &= \frac{r\sin(\theta)e^{-i\Omega}}{1 - 2r\cos(\theta)e^{-i\Omega} + r^2e^{-i2\Omega}} \end{aligned}$$

which can also be recognized as the response of a second order filter with the following recursion equation:

$$y[n] - 2r\cos(\theta)y[n-1] + r^2y[n-2] = r\sin(\theta)f[n-1] \qquad (19)$$

So once again we get the frequency response of a second order filter but this time by taking the imaginary part of the output of a first order filter. The normalized response is shown in figure 3 for $\theta = \pi/4$ and $r = [0.6, 0.7, 0.8, 0.9]$.
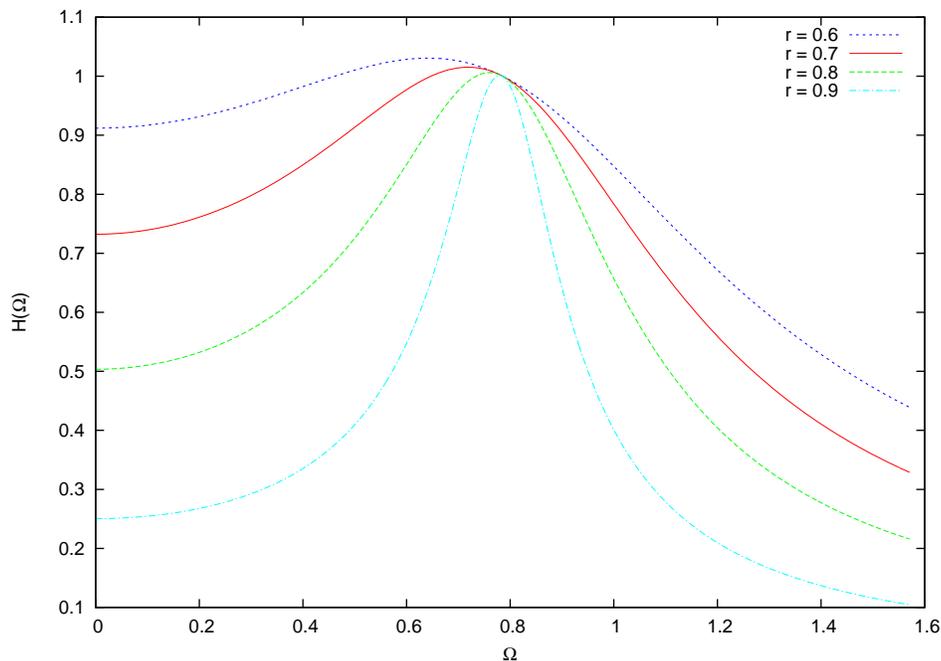
Figure 3: Eq. 18 normalized for $\theta = \pi/4$ and $r = [0.6, 0.7, 0.8, 0.9]$.

As before, the response is still that of a bandpass filter centered at $\Omega = \theta$ and with bandwidth controlled by the value of r. The peak magnitude used to normalize the response is:

$$|H(\theta)| = \left| \frac{r\sin(\theta)}{1-r} \right| \frac{1}{\sqrt{1 - 2r\cos(2\theta) + r^2}} \tag{20}$$

So far we have looked at two ways to get a real number from the output of our complex first order filter, we just use either the real or imaginary part of the output. We found that doing so, in both cases, actually gives us the frequency response of a second order filter with real coefficients. It may also have occurred to you that another way to get a real number is to just take the magnitude of the output. The problem with this is that you always get a positive number, so that what you have is a kind of digital rectifier which is probably not what you want.

Let's now look at what happens if we take the output of a first order filter with coefficient $c = re^{i\theta}$ and feed it into another such filter with coefficient $c^* = re^{-i\theta}$. If $w[n]$ is the output of the first filter then we have

$$Lw[n] = w[n] - cw[n-1] = f[n] \tag{21}$$

where the operator L has been introduced and defined. If $y[n]$ is the output of the second filter then

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$$L^*y[n] = y[n] - c^*y[n-1] = w[n] \tag{22}$$

Now to see how $y[n]$ is related to the input $f[n]$, just apply the operator L to this equation

$$
\begin{aligned}
LL^*y[n] &= Ly[n] - c^*Ly[n-1] = Lw[n] \\
&= y[n] - cy[n-1] - c^*(y[n-1] - cy[n-2]) = f[n]
\end{aligned} \tag{23}
$$

Substituting $c = re^{i\theta}$ into this expression and simplifying gives

$$y[n] - 2r\cos(\theta)y[n-1] + r^2y[n-2] = f[n] \tag{24}$$

This is a second order filter with frequency response

$$
\begin{aligned}
H(\Omega) &= \frac{1}{1 - 2r\cos(\theta)e^{-i\Omega} + r^2e^{-i2\Omega}} \\
&= \frac{1}{(1 - re^{i\theta}e^{-i\Omega})(1 - re^{-i\theta}e^{-i\Omega})}
\end{aligned} \tag{25}
$$

The second form of the response illustrates the fact that, when you feed the output of one filter into another, the overall response is the product of the responses of the two filters. This is once again a bandpass frequency response with peak magnitude occurring at $\Omega = \theta$ and bandwidth determined by r. The peak magnitude of eq. 25 is

$$|H(\theta)| = \frac{1}{|1 - r|\sqrt{1 - 2r\cos(2\theta) + r^2}} \tag{26}$$

The plot of this frequency response is identical to that shown in figure 3.

This now is about as far as you can go with these simple first order filters. To summarize, what we have found is that if you take two first order filters, with coefficients that are complex conjugates of one another, and combine them in various ways, you can produce second order bandpass filters. In the first case we simply took the real part of the output of one of these filters, which is equivalent to adding the outputs of two conjugate filters. We can call this a parallel filter combination. In the second case we took the imaginary part of the output, which is equivalent to subtracting the outputs of two conjugate filters. This is also a parallel filter combination. In the final case we feed the output of one filter into the input of a second filter which is a serial combination of the two filters. It is of course possible to combine more than two such first order filters. When combining three filters, two of them will have to be complex conjugates of one another and the third will have to be real in order for the final output to be real. Four filters would require two conjugate pairs and so on. Any general nth order recursive filter can be expressed as a parallel and/or serial combination of these first order filters which really does make them the "atoms" of recursive filters.